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Tangent bundle geometry for Lagrangian dynamics

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Abstract. Various aspects of the differential geometry of the tangent bundle of a differentiable manifold are examined, and the results applied to time-independent Lagrangian dynamics. It is shown that a certain type (1, 1) tensor field which is part of the intrinsic geometry of a tangent bundle, being a tensorial equivalent of the projection map of tangent vectors, plays a role in Lagrangian theory scarcely less important than that of the canonical one-form on a cotangent bundle in Hamiltonian theory. Recent results in Lagrangian theory are interpreted from this new viewpoint.

1. Introduction

The Hamiltonian picture of dynamics appears to be, in many ways, mathematically more straightforward and more elegant than the Lagrangian picture. This is because the cotangent bundle of the configuration space manifold of a mechanical system, which is the phase space of the system, carries a natural geometrical structure which is central to Hamiltonian theory, namely the canonical one-form $\theta = p_a \mathrm{d}q^a$, whose exterior derivative is the symplectic two-form on which the theory is based. Though Lagrangian theory, in the time-independent case, is also based on a symplectic structure, this time on the tangent bundle of configuration space, it is not a pre-existing structure but must be constructed out of the Lagrangian function. This makes for complexity. But recent research has uncovered some extremely interesting consequences of this complexity, such as the possibility of the existence of inequivalent alternative Lagrangian formulations of the same dynamical system, even in such a familiar case as a spherically symmetric potential in three dimensions (Henneaux and Shepley 1982); the generation of constants of the motion not derived from Noether-type theorems in these circumstances (Hojman and Harleston 1981); and the use of dynamical symmetries to create alternative Lagrangian functions (Prince 1983, Sarlet 1983). Much work has also been done recently on the inverse problem of Lagrangian dynamics. The complexity of this problem is convincingly demonstrated by Douglas's analysis of the two-dimensional case (Douglas 1941); of the many recent publications on this subject I shall single out for mention Henneaux's geometrical analysis (Henneaux 1982), Sarlet's detailed re-examination of the Helmholtz conditions (Sarlet 1982), and a paper of my own (Crampin 1981), since these references are particularly relevant to the present paper.

It has, however, been overlooked so far that the tangent bundle of a differential manifold does have naturally defined on it a simple geometrical object, which is loosely speaking dual to the canonical one-form of the cotangent bundle, and which plays a very interesting and important, if somewhat more subtle and hidden, role in the development of Lagrangian theory. This geometrical object is a type (1, 1) tensor field, which incorporates the projection map $\pi_*: T_{(q,u)}(T(M)) \to T_q(M)$ of tangent vectors into the intrinsic differential geometry of $T(M)$. (Here M is a differentiable manifold, $\pi: T(M) \to M$ its tangent bundle, (q, u) a point of $T(M)$ with $\pi(q, u) = q$.) It is dual to the canonical one-form in the sense that that object may be thought of as incorporating into the intrinsic differential geometry of the cotangent bundle $T^*(M)$ the pull-back map $\pi^*: T_q^*(M) \to T_{(q,p)}^*(T^*(M))$ of covectors. I shall show that this tensor field plays a key role in the analysis of the inverse problem of Lagrangian dynamics by use of horizontal distributions (Crampin 1981: it does actually make a brief appearance in that paper), and in the derivation of sequences of algebraic conditions (Henneaux 1982 and especially Sarlet 1982); it gives useful insights into the geometrical structure of the Euler–Lagrange equations and such classical notions as that of a variation; it is implicated in the derivation of the Hojman–Harleston result; and it provides for a geometrical formulation of the conditions found by Prince for a dynamical symmetry to generate an alternative Lagrangian.

I shall devote § 2 of the paper to demonstrating properties of the tensor field, having first reviewed various constructions of importance in tangent bundle geometry which are required for the definition. The following sections then develop Lagrangian theory in terms of the concepts of § 2; the arguments given in the main body of the paper are coordinate free, so various coordinate formulae are collected in § 7.

2. Geometry of the tangent bundle

The following constructions and results of tangent bundle geometry will be required. The standard reference is Yano and Ishihara (1973).

An element v of $T_{(q,u)}(T(M))$ satisfies $\pi_* v = 0$ (where $\pi: T(M) \to M$ is the projection) if and only if it is tangent to the fibre $\pi^{-1}(q)$. Such a vector is said to be vertical; the dim(M)-dimensional subspace of $T_{(q,u)}(T(M))$ consisting of vertical vectors is called the vertical subspace and a vector field V on $T(M)$ is said to be vertical if $V_{(q,u)}$ is vertical at each point (q, u). Any element ξ of $T_q(M)$ determines a vertical vector at any point (q, u) in the fibre over q, called its vertical lift to (q, u), denoted by $\xi^{\mathrm{v}}_{(q,u)}$; it is the tangent vector at $t = 0$ to the curve $t \mapsto u + t\xi$. The map $\xi \mapsto \xi^{\mathrm{v}}_{(q,u)}$ is an isomorphism of $T_q(M)$ with the vertical subspace of $T_{(q,u)}(T(M))$, which essentially duplicates the canonical isomorphism of a finite-dimensional real vector space with its tangent space at any point. The vertical lift X^{v} of a vector field X on M is the vertical vector field on $T(M)$ defined by $X^{\mathrm{v}}_{(q,u)} = (X_q)^{\mathrm{v}}_{(q,u)}$. It may be thought of as being constant along the fibres.

An important example of a vertical vector field which is not constant along the fibres of $T(M)$ is the dilation field Δ, which is the generator of the one-parameter group of dilations $\delta_t: (q, u) \mapsto (q, e^t u)$. A vector field W on $T(M)$ which satisfies $\delta_{t*} W = (e^t)^p W$, where p is an integer not less than -1, is homogeneous of degree p in the fibre coordinates, and satisfies $L_\Delta W = pW = [\Delta, W]$. A vertical lift is homogeneous of degree -1, and any vector field W such that $[\Delta, W] = -W$ must be the vertical lift of a vector field on M.

The Lie bracket of any two vertical vector fields is again vertical, and that of two vertical lifts is zero.

Any smooth map ϕ of M to itself lifts to a smooth map of $T(M)$ to itself by $(q, u) \mapsto (\phi(q), \phi_{*q}(u))$. If $\{\phi_t\}$ is a one-parameter group of transformations of M, with the vector field X as its generator, then the lifted transformations of $T(M)$ also form a one-parameter group, whose generator is a vector field on $T(M)$ called the complete lift of X and denoted by X^c. This construction applies, with the necessary modifications, to a vector field on M which generates only a flow which is not a one-parameter group. (On the few subsequent occasions in this paper on which I have to consider flows and one-parameter groups, I shall ignore the complications caused by the fact that not all vector fields are complete, that is, generate one-parameter groups. I trust that the reader can make the appropriate adjustments required to cope with incomplete vector fields.) Lie brackets involving complete lifts satisfy the following rules:

$$[X^v, Y^c] = [X, Y]^v \qquad [X^c, Y^c] = [X, Y]^c.$$

A complete lift is homogeneous of degree 0 in the fibre coordinates and so

$$[\Delta, X^c] = 0;$$

and in fact any vector field which commutes with Δ and is projectable (that is, π-related to a vector field on M) is a complete lift.

It is important to realise that, unlike the vertical lift, the complete lift does not have a pointwise equivalent: the complete lifts of two vector fields may differ at a point $(q, u) \in T(M)$ even though the vector fields themselves agree at $q \in M$.

If $\{X_1, X_2, \ldots, X_m\}$ is a local basis of vector fields on M, where $m = \dim(M)$, then $\{X_1^c, X_2^c, \ldots, X_m^c, X_1^v, X_2^v, \ldots, X_m^v\}$ is a local basis of vector fields on $T(M)$. It is therefore sufficient, in order to determine uniquely a tensor field on $T(M)$, to specify its action on the complete and vertical lifts of vector fields on M.

Any curve σ in M has a natural lift to $T(M)$, namely the curve $t \mapsto (\sigma(t), \dot{\sigma}(t))$, where $\dot{\sigma}(t)$ is the tangent vector to σ at $\sigma(t)$. A vector field on $T(M)$ whose integral curves are all natural lifts of curves on M in this sense is called a second-order differential equation field. This terminology is justified by the fact that the projections of its integral curves onto M, when expressed in terms of coordinates, are the solutions of a system of second-order ordinary differential equations (see § 7 for the details). A vector field Γ on $T(M)$ is a second-order differential equation field if and only if it satisfies the condition

$$\pi_* \Gamma_{(q,u)} = u \qquad \text{for all } (q, u) \in T(M).$$

The final introductory point concerns notation. I shall use i_X to denote the interior product of a form by the vector field X, so that for example if Ω is a two-form, $i_X \Omega$ is the one-form defined by

$$(i_X \Omega)(Y) = \Omega(X, Y).$$

I shall have to deal on several occasions with a somewhat similar construction involving a two-form and a type (1, 1) tensor field. If T is a type (1, 1) tensor field and Ω a two-form then

$$(X, Y) \mapsto \Omega(T(X), Y)$$

defines a type (0, 2) tensor field, not necessarily skew-symmetric (it would be described in tensor calculus as the tensor obtained from T by using Ω to lower the contravariant index). I shall denote this type (0, 2) tensor field by $T \lrcorner \Omega$. It is easy to show that for any vector field X,

$$L_X(T \lrcorner \Omega) = (L_X T) \lrcorner \Omega + T \lrcorner L_X \Omega.$$

The principal concern of this memoir is to derive and exploit the properties of the tensor field S, of type (1, 1), whose action as a linear map of a tangent space to $T(M)$ is the composition of projection and vertical lift: if $v \in T_{(q,u)}(T(M))$ then

$$S_{(q,u)}(v) = (\pi_* v)^{\mathrm{v}}_{(q,u)}.$$

In terms of complete and vertical lifts,

$$S(X^{\mathrm{c}}) = X^{\mathrm{v}} \qquad S(X^{\mathrm{v}}) = 0.$$

The tensor field S is the vertical lift of the identity tensor field (Kronecker δ) on M to $T(M)$, in the sense of Yano and Ishihara (1973, p 34).

I next give some of the purely geometric properties of S. Evidently $S^2 = S \circ S = 0$. The necessary and sufficient condition for a vector field W on $T(M)$ to be vertical is that $S(W) = 0$, and for it to be projectable is that $S(W)$ is a vertical lift. The Lie derivatives of S by vertical and complete lifts may be computed by using the formula

$$(L_{W_1}S)(W_2) = [W_1, S(W_2)] - S([W_1, W_2]).$$

Thus (for X_1, X_2 vector fields on M)

$$(L_{X_1^{\mathrm{v}}}S)(X_2^{\mathrm{c}}) = [X_1^{\mathrm{v}}, X_2^{\mathrm{v}}] - S([X_1, X_2]^{\mathrm{v}}) = 0 \qquad (L_{X_1^{\mathrm{v}}}S)(X_2^{\mathrm{v}}) = 0$$

$$(L_{X_1^{\mathrm{c}}}S)(X_1^{\mathrm{c}}) = [X_1, X_2]^{\mathrm{v}} - S([X_1, X_2]^{\mathrm{c}}) = 0 \qquad (L_{X_1^{\mathrm{c}}}S)(X_2^{\mathrm{v}}) = -S([X_1, X_2]^{\mathrm{v}}) = 0.$$

Therefore, the Lie derivative of S by either a vertical or a complete lift vanishes. It is easy to show by the same method that

$$L_\Delta S = -S.$$

Next, I show that the Nijenhuis tensor of S is zero. The Nijenhuis tensor N_T of a type (1, 1) tensor field T is the type (1, 2) tensor field defined by

$$N_T(X_1, X_2) = [T(X_1), T(X_2)] + T^2([X_1, X_2]) - T([T(X_1), X_2]) - T([X_1, T(X_2)])$$

where X_1 and X_2 are any two vector fields. In the case of interest, since $S^2 = 0$, for any vector fields W_1, W_2 on $T(M)$,

$$N_S(W_1, W_2) = [S(W_1), S(W_2)] - S([S(W_1), W_2]) - S([W_1, S(W_2)]).$$

Moreover, since N_S is evidently skew-symmetric, it is enough to show that N_S vanishes when its arguments are respectively a pair of complete lifts; a vertical lift and a complete lift; a pair of vertical lifts. But

$$\begin{aligned} N_S(X_1^{\mathrm{c}}, X_2^{\mathrm{c}}) &= [X_1^{\mathrm{v}}, X_2^{\mathrm{v}}] - S([X_1^{\mathrm{v}}, X_2^{\mathrm{c}}]) - S([X_1^{\mathrm{c}}, X_2^{\mathrm{v}}]) \\ &= -S([X_1, X_2]^{\mathrm{v}}) + S([X_1, X_2]^{\mathrm{v}}) = 0 \\ N_S(X_1^{\mathrm{v}}, X_2^{\mathrm{c}}) &= -S([X_1^{\mathrm{v}}, X_2^{\mathrm{v}}]) = 0 \qquad N_S(X_1^{\mathrm{v}}, X_2^{\mathrm{v}}) = 0 \end{aligned}$$

for any vector fields X_1, X_2 on M.

A type (1, 1) tensor may be applied to one-forms as well as to vector fields; if one-forms are regarded as maps of vector fields to functions, the action of a type (1, 1) tensor is simply composition. If θ is a one-form on $T(M)$ then $\theta \circ S$ vanishes on vertical vector fields.

From a two-form ω one may construct two type (0, 2) tensor fields using S: the field $S \lrcorner \omega$ mentioned earlier, and the field $(W_1, W_2) \mapsto \omega(S(W_1), S(W_2))$. In the case of an exact two-form $\omega = \mathrm{d}\theta$, these three constructions are related as follows: for any vector fields W_1, W_2 on $T(M)$,

$$d\theta(S(W_1), S(W_2)) = d(\theta \circ S)(S(W_1), W_2) + d(\theta \circ S)(W_1, S(W_2));$$

that is, $d\theta(S(\cdot), S(\cdot))$ is, but for a constant factor, the skew-symmetric part of $S \lrcorner d(\theta \circ S)$. In fact

$$d(\theta \circ S)(S(W_1), W_2) = S(W_1)(\theta(S(W_2))) - \theta(S([S(W_1), W_2]))$$

$$d(\theta \circ S)(W_1, S(W_2)) = -S(W_2)(\theta(S(W_1))) - \theta(S([W_1, S(W_2)]))$$

and so, as a consequence of the vanishing of the Nijenhuis tensor of S,

$$\begin{aligned} d(\theta \circ S)(S(W_1), W_2) + d(\theta \circ S)(W_1, S(W_2)) \\ &= S(W_1)(\theta(S(W_2))) - S(W_2)(\theta(S(W_1))) - \theta([S(W_1), S(W_2)]) \\ &= d\theta(S(W_1), S(W_2)) \end{aligned}$$

as required. In particular, if θ is closed then $S \lrcorner d(\theta \circ S)$ is symmetric, which may equivalently be stated in the form

$$i_{S(W)}\omega + (i_W\omega) \circ S = 0 \qquad (\omega = d(\theta \circ S))$$

for any vector field W.

I now turn to some results relating S and the properties of second-order differential equation fields. A vector field Γ on $T(M)$ is a second-order differential equation field if and only if

$$S(\Gamma) = \Delta.$$

It follows that for any second-order differential equation field Γ and any vector field X on M, $[X^c, \Gamma]$ is vertical and $[X^v, \Gamma]$ projects onto X: for

$$S([X^c, \Gamma]) = [X^c, S(\Gamma)] = [X^c, \Delta] = 0 \qquad S([X^v, \Gamma]) = [X^v, S(\Gamma)] = [X^v, \Delta] = X^v.$$

The key result concerns the Lie derivative $L_\Gamma S$ of S by any second-order differential equation field: this tensor field satisfies

$$(L_\Gamma S)^2 = I$$

where I is the identity type $(1, 1)$ tensor field on $T(M)$. This I now prove. Firstly, for any vector field X on M,

$$(L_\Gamma S)(X^v) = -S([\Gamma, X^v]) = X^v.$$

It follows that $L_\Gamma S$ acts as the identity on all vertical vector fields; and evidently so does $(L_\Gamma S)^2$. On the other hand,

$$(L_\Gamma S)(X^c) = [\Gamma, X^v] - S([\Gamma, X^c]) = [\Gamma, X^v]$$

since $[\Gamma, X^c]$ is vertical. Now $[\Gamma, X^v]$ projects on $-X$ while X^c projects on X, and so $(L_\Gamma S)(X^c) + X^c$ is vertical. Thus

$$(L_\Gamma S)((L_\Gamma S)(X^c) + X^c) = (L_\Gamma S)(\dot{X}^c) + X^c$$

from which it follows that

$$(L_\Gamma S)^2(X^c) = X^c$$

and the proof is complete.

The tensor fields P and Q on $T(M)$ given by

$$P = \tfrac{1}{2}(I - L_\Gamma S) \qquad Q = \tfrac{1}{2}(I + L_\Gamma S)$$

are therefore projection operators:

$$P^2 = P \qquad Q^2 = Q$$

such that

$$P \circ Q = Q \circ P = 0.$$

Two distributions on $T(M)$ are defined by P and Q: $\ker P$, the set of vector fields on which P vanishes, which is equivalently defined as $\operatorname{im} Q$, the set of vector fields of the form $Q(\cdot)$; and $\ker Q$, or equivalently $\operatorname{im} P$. These distributions could also be defined as those on which $L_\Gamma S$ acts as the identity, and minus the identity, respectively. At each point of $T(M)$ the tangent space is the direct sum of complementary subspaces defined by the two distributions. Since $L_\Gamma S$ acts as the identity on vertical vector fields, $\ker P$ contains all vertical vector fields. On the other hand, if $W \in \ker P$ then $(L_\Gamma S)(W) = W$, and therefore $S(W) = S((L_\Gamma S)(W))$; but $S^2 = 0$, and therefore $S \circ L_\Gamma S = -L_\Gamma S \circ S$; thus $S(W) = -(L_\Gamma S)(S(W)) = -S(W)$ since $S(W)$ is vertical; consequently $S(W) = 0$, and W is vertical. Thus $\ker P$ consists precisely of the vertical distribution. It follows that $\ker Q$ ($\operatorname{im} P$) is a distribution of the same dimension as the vertical distribution (namely $\dim(M)$), which, since it is complementary to the vertical distribution, will be called horizontal. Moreover, for any vector field X and smooth function f on M, the vector field $(fX)^{\mathrm{c}}$ differs from $(\pi^* f)X^{\mathrm{c}}$ by a vertical vector field, and therefore $P((fX)^{\mathrm{c}}) = (\pi^* f)P(X^{\mathrm{c}})$; it follows that every vector field X on M has a horizontal lift X^{h} to $T(M)$ defined by $X^{\mathrm{h}} = P(X^{\mathrm{c}})$, the map from vector fields on M to horizontal vector fields on $T(M)$ so defined being linear over the ring of smooth functions on M. Consequently, the value $X^{\mathrm{h}}_{(q,u)}$ of the horizontal lift of X at $(q, u) \in T(M)$ depends only on the value X_q of X at q (and not on its values at other points of M), and the map $\xi \mapsto \xi^{\mathrm{h}}_{(q,u)}$, where $\xi \in T_q(M)$ and $\xi^{\mathrm{h}}_{(q,u)}$ is the value at (q, u) of the horizontal lift of any vector field on M whose value at q is ξ, is a linear one. This linear map is in fact an isomorphism of $T_q(M)$ with the horizontal subspace of $T_{(q,u)}(T(M))$, whose inverse is simply the projection restricted to the horizontal subspace.

An explicit formula for X^{h}, in terms of the particular second-order differential equation field Γ under discussion, is given by

$$X^{\mathrm{h}} = P(X^{\mathrm{c}}) = \tfrac{1}{2}(X^{\mathrm{c}} - (L_\Gamma S)(X^{\mathrm{c}})) = \tfrac{1}{2}(X^{\mathrm{c}} + [X^{\mathrm{v}}, \Gamma]).$$

This construction generalises the horizontal distribution of a symmetric linear connection on M, to which it reduces when Γ is the 'spray' determined by the connection, that is, its geodesic field. The explicit formula for a horizontal lift has been given before (Crampin 1971, 1981; see also Yano and Ishihara 1973, ch VI), and the extension of some ideas of connection theory to horizontal distributions in general is the subject of a further paper by the present author (Crampin 1983a).

3. Lagrangian theory

From any function L on $T(M)$ (a Lagrangian function) one may construct a two-form ω by

$$\omega = \mathrm{d}(\mathrm{d}L \circ S).$$

The Lagrangian L is said to be regular if the two-form ω has maximal rank, that is, if

$$i_W\omega = 0 \qquad \text{implies that } W = 0.$$

Since $\omega = \mathrm{d}(\theta \circ S)$ and the one-form $\theta = \mathrm{d}L$ is certainly closed, it follows from the vanishing of the Nijenhuis tensor of S that for every pair of vector fields W_1, W_2 on $T(M)$

$$\omega(S(W_1), W_2) + \omega(W_1, S(W_2)) = 0;$$

in other words, the type (0, 2) tensor field $S \lrcorner \omega$ is symmetric. Thus for any vector field W,

$$i_{S(W)}\omega = -(i_W\omega) \circ S.$$

The two-form ω is given explicitly by

$$\omega(W_1, W_2) = W_1(S(W_2)(L)) - W_2(S(W_1)(L)) - S([W_1, W_2])(L),$$

for any vector fields W_1, W_2 on $T(M)$. When Δ is taken for W_1, one finds that

$$\begin{aligned}\omega(\Delta, W) &= \Delta(S(W)(L)) - S([\Delta, W])(L)\\ &= S(W)(\Delta(L)) + ([\Delta, S(W)] - S([\Delta, W]))(L)\\ &= S(W)(\Delta(L)) + ((L_\Delta S)(W))(L)\\ &= S(W)(\Delta(L)) - S(W)(L)\\ &= S(W)(E)\end{aligned}$$

where $E = \Delta(L) - L$ is the energy function associated with L. Thus

$$i_\Delta\omega = \mathrm{d}E \circ S.$$

If Γ is a second-order differential equation field, then

$$(i_\Gamma\omega) \circ S = -i_{S(\Gamma)}\omega = -\mathrm{d}E \circ S$$

so that $i_\Gamma\omega + \mathrm{d}E$ vanishes on vertical vector fields. Conversely, if W is any vector field such that $i_W\omega + \mathrm{d}E$ vanishes on vertical vector fields then

$$(i_W\omega + \mathrm{d}E) \circ S = 0 = -i_{S(W)}\omega + i_\Delta\omega$$

whence $S(W) = \Delta$ (assuming that L is regular) and W is a second-order differential equation field.

There is a unique field Λ such that

$$i_\Lambda\omega = -\mathrm{d}E$$

and by the argument above it is a second-order differential equation field. It is the Euler–Lagrange field for L, since the second-order differential equations satisfied by its integral curves are the Euler–Lagrange equations for L.

The explicit equation for ω gives, when one of the arguments is an arbitrary second-order differential equation field Γ,

$$\omega(\Gamma, W) = \Gamma(S(W)(L)) - W(\Delta(L)) - S([\Gamma, W])(L)$$

which may be rearranged into the form

$$W(L) = -(i_\Gamma\omega + \mathrm{d}E)(W) + \Gamma(S(W)(L)) + S([W, \Gamma])(L).$$

Suppose, now, that $S([W,\Gamma])=0$, in other words, that $L_W\Gamma$ is vertical. Then W generates a one-parameter group $\{\phi_t\}$ such that for every t, $\phi_{t*}\Gamma$ differs from Γ by a vector field which is, at least to first order in t, vertical; and so $\phi_{t*}\Gamma$ is (again to first order in t) a second-order differential equation field. Since time, in the present case, is represented by the parameter on the integral curves of the vector field representing the dynamics, ϕ_t may be regarded as a 'variation by which we pass from a point of (the original curve) to a point of (the transformed curve) which is correlated to the same value of the time' (Whittaker 1904). A vector field W such that $S([W,\Gamma])=0$ will therefore be called a variation of the second-order differential equation field Γ.

A complete lift is a variation field of any second-order differential field; in fact, for any second-order differential equation field Γ and any vector field X on M there is a unique variation field W of Γ such that $S(W)=X^{\mathrm{v}}$, namely $W=X^{\mathrm{c}}$. More general variation fields are possible, though no longer universally: as I shall now show, given any vertical vector field V there is a unique variation field V_Γ of Γ such that $S(V_\Gamma)=V$; as the notation indicates, the variation field depends on Γ. The proof proceeds by showing that the map $W\mapsto S(W)$ of variation fields of Γ, which is linear over R, is bijective. If W is a variation of Γ such that $S(W)=0$, so that W is vertical, then

$$W=(L_\Gamma S)(W)=[\Gamma,S(W)]-S([\Gamma,W])=0.$$

The map is thus injective. On the other hand, if V is a vertical vector field, set $V_\Gamma=(L_\Gamma S)([\Gamma,V])$; then

$$S(V_\Gamma)=(S\circ L_\Gamma S)([\Gamma,V])=-(L_\Gamma S)(S([\Gamma,V]))=(L_\Gamma S)^2(V)=V$$

and

$$S([V_\Gamma,\Gamma])=(L_\Gamma S)(V_\Gamma)-[\Gamma,V]=(L_\Gamma S)^2([\Gamma,V])-[\Gamma,V]=0.$$

Thus V_Γ is a variation of Γ such that $S(V_\Gamma)=V$, and the map is also surjective.

It follows that for any second-order differential equation field Γ and for any vertical vector field V,

$$V_\Gamma(L)=-(i_\Gamma\omega+\mathrm{d}E)(V_\Gamma)+\Gamma(V(L)).$$

This is a global version of the variational equation of Lagrangian dynamics. By specialising it to any integral curve of Γ, integrating along an interval of the curve, and applying the usual arguments of the calculus of variations (as set out for example in Gelfand and Fomin (1963)), one obtains the Euler–Lagrange equations in the form noted above, $i_\Lambda\omega=-\mathrm{d}E$. It follows from the Euler–Lagrange equations in this form that

$$L_\Lambda\omega=0.$$

I now return to the fact that $S\lrcorner\omega$ is symmetric and describe some of its consequences. Since

$$L_\Lambda(S\lrcorner\omega)=(L_\Lambda S)\lrcorner\omega$$

and since the Lie derivative of a symmetric tensor is symmetric, $(L_\Lambda S)\lrcorner\omega$ is also symmetric. It follows that for any pair of vector fields W_1, W_2,

$$\omega(P(W_1),W_2)+\omega(W_1,P(W_2))=\omega(W_1,W_2)$$

$$\omega(Q(W_1),W_2)+\omega(W_1,Q(W_2))=\omega(W_1,W_2)$$

$$\omega(P(W_1),W_2)-\omega(W_1,Q(W_2))=0.$$

As a consequence of the first two of these relations one finds that ω vanishes when both its arguments are horizontal, and when both its arguments are vertical; that is to say, the horizontal and vertical subspaces of the tangent space at each point of $T(M)$ are both Lagrangian for ω.

The simple argument used above to derive the symmetry of $(L_\Lambda S)\lrcorner\omega$ from that of $S\lrcorner\omega$ applies quite generally to show that if, for any tensor T, $T\lrcorner\omega$ is symmetric, then $(L_\Lambda T)\lrcorner\omega$ is also symmetric. Furthermore, if $T\lrcorner\omega$ is symmetric then so are $(P\circ T\circ Q)\lrcorner\omega$ and $(Q\circ T\circ P)\lrcorner\omega$; for (to deal only with the first case, the second being essentially the same),

$$\omega(P(T(Q(W_1))), W_2)=\omega(T(Q(W_1)), Q(W_2))$$
$$=-\omega(Q(W_1), T(Q(W_2)))=-\omega(W_1, P(T(Q(W_2))))$$

as required. The generation of symmetric tensors by combining the two processes of taking the Lie derivative and composing with P and Q lies at the basis of the derivation of a sequence of algebraic necessary conditions for the existence of a Lagrangian (Henneaux 1982, Sarlet 1982), a point to which I return in § 5.

Finally, since $L_\Lambda S$ is non-singular, the symmetric tensor $(L_\Lambda S)\lrcorner\omega$ defines a metric on $T(M)$. It generalises the so-called complete lift to $T(M)$ of a metric on M, to which it reduces when L is the kinetic energy Lagrangian for such a metric and Λ is therefore the geodesic spray of the corresponding Levi-Cività connection. The metric $(L_\Lambda S)\lrcorner\omega$ is pseudo-Riemannian, with dim(M) positive and dim(M) negative signs in its signature, the vertical and horizontal subspaces at each point being null. (It should be noted that there is more than one way of associating a metric with ω, and this one differs from that given in Crampin (1981).)

4. The inverse problem of Lagrangian dynamics

Given a second-order differential equation field on the tangent bundle of a differentiable manifold, one may ask: under what conditions is it the Euler-Lagrange field for some Lagrangian function (to be determined)? Conditions for the existence of a 'multiplier' for a system of second-order differential equations $\ddot{q}^a=f^a(q,\dot{q})$, that is to say, a matrix (α_{ab}) of functions of $q, \dot{q}$ such that the equations $\alpha_{ab}\ddot{q}^b+\beta_a=0$ (where $\beta_a=-\alpha_{ab}f^b$) are equivalent to the original ones and of the Euler-Lagrange form, have been known for a long time, usually being attributed to Helmholtz (see e.g. Santilli 1978). The Helmholtz conditions provide only a partial solution to the problem, since they are conditions to be satisfied by the multiplier (α_{ab}), whereas ideally one would look for conditions which could be stated in terms of the functions f^a alone. However, the analysis of the two-dimensional case by Douglas (1941) gives a clear indication that this ideal is not likely to be attained in a general way. Recent investigations of the problem have achieved improvements over the Helmholtz conditions in their original form, by replacing some of the differential conditions on the multiplier matrix by algebraic conditions involving, essentially, higher and higher derivatives of the functions f^a. In particular, Henneaux (1982), using a Lie derivative based approach, gave a sequence of algebraic necessary conditions and Sarlet (1982), by purely analytic methods, obtained a set of necessary and sufficient conditions which also include a sequence of algebraic conditions. I wish to show how, by adapting Henneaux's methods, one may obtain Sarlet's conditions, at least in the time-independent case.

As I suggested above, once again the tensor field S and its Lie derivatives will play a key role.

I have shown elsewhere (Crampin 1981) that the Helmholtz conditions may be equivalently expressed in geometrical language as follows: necessary and sufficient conditions for a second-order differential equation field Γ to be derivable from a regular Lagrangian function are the existence of a two-form ω, of maximal rank, for which $L_\Gamma\omega = 0$, and such that all vertical subspaces are Lagrangian both for ω and for $i_H d\omega$ where H is any horizontal vector field (horizontal, that is, with respect to the horizontal distribution determined by Γ). It follows, of course, from these assumptions about ω that $S \lrcorner \omega$ is symmetric: this may be shown as follows. It is certainly true that

$$\omega(S(W_1), W_2) + \omega(W_1, S(W_2)) = 0$$

when W_1 and W_2 are both vertical; and when W_1 (say) is horizontal and W_2 vertical it is a consequence of the assumption that vertical subspaces are Lagrangian. Since $L_\Gamma\omega = 0$, for any vector fields X_1, X_2 on M,

$$\omega([\Gamma, X_1^{\mathrm{v}}], X_2^{\mathrm{v}}) + \omega(X_1^{\mathrm{v}}, [\Gamma, X_2^{\mathrm{v}}]) = 0;$$

the horizontal component of $[\Gamma, X^{\mathrm{v}}]$, which is the only one that matters, is $-X^{\mathrm{h}}$, whence

$$\omega(X_1^{\mathrm{h}}, X_2^{\mathrm{v}}) + \omega(X_1^{\mathrm{v}}, X_2^{\mathrm{h}}) = 0 = \omega(X_1^{\mathrm{h}}, S(X_2^{\mathrm{h}})) + \omega(S(X_1^{\mathrm{h}}), X_2^{\mathrm{h}}),$$

which completes the proof.

It now follows that tensor fields derived from S by repeated Lie differentiation with respect to Γ and by composition with P and Q, in the manner given earlier, have the same symmetry property with respect to ω as does S itself. To be explicit, the sequence of type (1, 1) tensor fields $\Phi^{(k)}$, $k = 0, 1, 2, \ldots$ defined by

$$\Phi^{(k+1)} = Q \circ L_\Gamma \Phi^{(k)} \circ P \qquad \Phi^{(0)} = -\tfrac{1}{2} Q \circ L_\Gamma(L_\Gamma S) \circ P$$

is such that $\Phi^{(k)} \lrcorner \omega$ is symmetric for every k. The tensor field $\Phi^{(k)}$ is of course defined in terms of the intrinsic geometry of the tangent bundle and of the particular second-order differential equation field Γ, and is not dependent on the existence of a multiplier. The symmetry conditions are therefore algebraic conditions on ω, whose coefficients are simply related to the multiplier. These conditions are essentially equivalent to the algebraic conditions given by Sarlet, making allowance for the fact that he was dealing with the time-dependent case, as will be shown in § 7 by exhibiting the coordinate expressions for the $\Phi^{(k)}$; the coefficients and the numbering of the $\Phi^{(k)}$ have been chosen to agree with his. Of course, one could define sequences of symmetric tensors based on S in other ways; the significance of the chosen one is that composition with Q and P picks out the component containing the highest derivative at each stage. In other words, though $L_\Gamma\Phi^{(k)} \lrcorner \omega$ is certainly symmetric, the terms in $L_\Gamma\Phi^{(k)}$ which are not in $Q \circ L_\Gamma\Phi^{(k)} \circ P$ turn out to satisfy the symmetry condition automatically since they depend on the $\Phi^{(l)}$ with $l \leq k$, whose symmetry is already established.

Sarlet, in fact, went further and showed in effect that the following conditions are necessary and sufficient for the existence of a Lagrangian function for the second-order differential equation field Γ: the existence of a two-form ω of maximal rank such that $S \lrcorner \omega$, $(L_\Gamma S) \lrcorner \omega$ and $\Phi^{(k)} \lrcorner \omega$, $k = 0, 1, 2, \ldots$ are all symmetric, and that $S \lrcorner (i_H d\omega)$ is symmetric for every horizontal vector field H. Thus he was able to trade off the assumption that $L_\Gamma\omega = 0$ against the sequence of algebraic conditions and the weakening of the conditions on the exterior derivative. This is an ingenious and somewhat unexpected achievement.

5. Alternative Lagrangians

When there are two Lagrangian functions for the same second-order differential equation field Γ which are not just related by the trivial operations of multiplication by a constant and addition of a total time derivative, then the corresponding two-forms ω and $\tilde{\omega}$ both satisfy the geometrical version of the Helmholtz conditions quoted above; and conversely, two distinct two-forms, not differing by a constant factor, both satisfying the conditions define alternative not trivially equivalent Lagrangians for Γ. In these circumstances it follows from the fact that ω and $\tilde{\omega}$ are of maximal rank that there is a non-singular type (1, 1) tensor field R such that

$$\tilde{\omega} = R \lrcorner \omega.$$

The tensor field R must satisfy a number of algebraic and differential conditions arising from those assumed for ω and $\tilde{\omega}$, as follows. Firstly, if T is any type (1, 1) tensor field such that $T \lrcorner \omega$ and $T \lrcorner \tilde{\omega}$ are symmetric, then R commutes with T and $(R \circ T) \lrcorner \omega$ is also symmetric. For

$$\tilde{\omega}(T(W_1), W_2) = \tilde{\omega}(T(W_2), W_1) = -\tilde{\omega}(W_1, T(W_2))$$

for any vector fields W_1, W_2, from which it follows that

$$\omega(R(T(W_1)), W_2) = -\omega(R(W_1), T(W_2)) = \omega(T(R(W_1)), W_2)$$

so that $R \circ T = T \circ R$. Furthermore, by the skew-symmetry of $\tilde{\omega}$,

$$\omega(R(T(W_1)), W_2) = -\omega(R(W_2), T(W_1)) = \omega(T(R(W_2)), W_1)$$

whence $(R \circ T) \lrcorner \omega$ is symmetric. In particular R commutes with $L_\Gamma S$, and therefore with the projection operators P and Q; it therefore preserves the direct sum decomposition of tangent spaces into horizontal and vertical subspaces, mapping horizontal to horizontal vectors and vertical to vertical. Moreover, from the fact that R commutes with S it follows that for any vector field X on M,

$$S(R(X^{\mathrm{h}})) = R(X^{\mathrm{v}}),$$

so that R acts essentially identically on horizontal and on vertical lifts. To be precise, if $\{X_1, X_2, \ldots, X_m\}$ is a local basis of vector fields on M, with horizontal lifts $\{H_1, H_2, \ldots, H_m\}$ and vertical lifts $\{V_1, V_2, \ldots, V_m\}$, then with respect to the local basis $\{H_1, H_2, \ldots, H_m, V_1, V_2, \ldots, V_m\}$ of vector fields on $T(M)$, R is represented by a $2m \times 2m$ matrix with identical $m \times m$ blocks on the diagonal and zeros elsewhere.

More generally, R must commute with $\Phi^{(k)}$, $k = 0, 1, 2, \ldots$. This gives a sequence of necessary algebraic conditions that R must satisfy, which are useful in the investigation of the existence of alternative Lagrangians for a dynamical system for which one Lagrangian is already known, a technique exploited by Henneaux and Shepley (1982).

From the assumption that $L_\Gamma\omega = L_\Gamma\tilde{\omega} = 0$ it follows that $L_\Gamma R = 0$, and that consequently $L_\Gamma R^k = 0$ for every power R^k of R. Moreover, since Lie differentiation commutes with contraction, it follows that the trace of R and of each of its powers is a constant of the motion, the result of Hojman and Harleston (1981). This is a particular case of a general geometrical construction for finding constants of the motion

(Crampin 1983b). A related proof of the Hojman–Harleston result, which does not however use the technique of vertical and horizontal decomposition, has been given by Henneaux (1981). It may be shown, using the analysis presented above, and recent results of de Filippo *et al* (1983), that if a dynamical system admits alternative Lagrangians, and the tensor field R has vanishing Nijenhuis tensor, then the system is completely integrable (Crampin *et al* 1983).

Finally, from the closure condition for ω and $\tilde{\omega}$ it follows that the vertical Lie derivatives of R satisfy

$$\omega((L_{Y^{\mathrm{v}}}R)(X^{\mathrm{h}}), Z^{\mathrm{v}}) = \omega((L_{Z^{\mathrm{v}}}R)(X^{\mathrm{h}}), Y^{\mathrm{v}})$$

for all vector fields X, Y, Z on M. This is a version of the symmetry condition on vertical derivatives of the 'Hessian matrix' given by Henneaux and Shepley.

6. Symmetries

In discussing symmetries of an Euler–Lagrange field, or more generally of a second-order differential equation field, I shall follow the classification of Prince (1983), making the necessary allowances for the fact that he deals with the time-dependent case.

A Lie symmetry of a second-order differential equation field Γ on $T(M)$ is a vector field X on M such that $[X^{\mathrm{c}}, \Gamma] = 0$. (A direct transliteration of Prince's definition of a Lie symmetry would apparently allow $[X^{\mathrm{c}}, \Gamma]$ to be a multiple of Γ, but since $[X^{\mathrm{c}}, \Gamma]$ is vertical the factor must be zero.) Thus the one-parameter group of transformations of $T(M)$ generated by X^{c} permutes the integral curves of Γ. In the case of an Euler–Lagrange field Λ, a vector field X on M such that $L_{X^{\mathrm{c}}}(\mathrm{d}L \circ S)$ is exact and $X^{\mathrm{c}}(E)$ is zero is a Lie symmetry of Λ, since

$$i_{[X^{\mathrm{c}},\Lambda]}\omega = -L_{X^{\mathrm{c}}}(\mathrm{d}E) - i_{\Lambda}(L_{X^{\mathrm{c}}}\omega) = 0;$$

such a symmetry is called a Noether symmetry. Noether symmetries give rise to constants of the motion. This can be seen in two ways. Firstly, using the variational equation, and the fact that X^{c} is the variation field corresponding to the vertical vector field X^{v} for any second-order differential equation field Γ, one finds that

$$X^{\mathrm{c}}(L) = -(i_\Gamma\omega + \mathrm{d}E)(X^{\mathrm{c}}) + \Gamma(X^{\mathrm{v}}(L)).$$

If now $L_{X^{\mathrm{c}}}(\mathrm{d}L \circ S) = \mathrm{d}f$ then, since $L_{X^{\mathrm{c}}}S = 0$,

$$\mathrm{d}f = \mathrm{d}(X^{\mathrm{c}}(L)) \circ S$$

and therefore

$$\Gamma(f) = \Delta(X^{\mathrm{c}}(L)) = X^{\mathrm{c}}(\Delta(L)) = X^{\mathrm{c}}(L)$$

since $[\Delta, X^{\mathrm{c}}] = 0$ and $X^{\mathrm{c}}(E) = 0$. The variational equation now reads

$$\Gamma(f - X^{\mathrm{v}}(L)) = -(i_\Gamma\omega + \mathrm{d}E)(X^{\mathrm{c}})$$

and when $\Gamma = \Lambda$, the Euler–Lagrange field, one obtains

$$\Lambda(f - X^{\mathrm{v}}(L)) = 0.$$

This derivation is a version of Noether's theorem. The key step is that $\Gamma(f) = X^{\mathrm{c}}(L)$ for every second-order differential equation field Γ. The second approach, which could be called the Cartan approach, is more direct. From the condition $L_{X^c}(\mathrm{d}L \circ S) = \mathrm{d}f$ one finds that

$$i_{X^c}\omega + \mathrm{d}(i_{X^c}(\mathrm{d}L \circ S)) = i_{X^c}\omega + \mathrm{d}(X^{\mathrm{v}}(L)) = \mathrm{d}f$$

and therefore

$$\Lambda(f - X^{\mathrm{v}}(L)) = (i_{X^c}\omega)(\Lambda) = -(i_\Lambda\omega)(X^c) = X^c(E) = 0.$$

I give these alternatives because they are instructive models for the more general type of symmetry which is not simply generated by a vector field on the base manifold M. A dynamical symmetry of a second-order differential equation field Γ is a vector field W on $T(M)$ such that $[W, \Gamma] = 0$. A Cartan symmetry of an Euler–Lagrange field Λ is a vector field W on $T(M)$ such that $L_W(\mathrm{d}L \circ S)$ is exact and $W(E) = 0$. A Cartan symmetry is a special case of a dynamical symmetry of Λ as before. Moreover, the Cartan argument above applies with little alteration to show that if $L_W(\mathrm{d}L \circ S) = \mathrm{d}f$ then $f - S(W)(L)$ is a constant of the motion:

$$i_W\omega + \mathrm{d}(S(W)(L)) = \mathrm{d}f \qquad \Lambda(f - S(W)(L)) = -(i_\Lambda\omega)(W) = W(E) = 0.$$

The oddity of the Noether argument in this case is that the variation field W_Γ which is required in the variational equation, namely the variation field of Γ such that $S(W_\Gamma) = S(W)$, will vary with Γ; only when $\Gamma = \Lambda$ will $W_\Gamma = W$. Though the Noether argument does follow through, the Cartan argument is clearly the more straightforward.

Lie (Noether) symmetries may be characterised as those dynamical (Cartan) symmetries for which W is projectable. In fact if $[W, \Gamma] = 0$ and $S(W) = X^{\mathrm{v}}$, then $(L_\Gamma S)(W) = [\Gamma, X^{\mathrm{v}}]$, and so $W = (L_\Gamma S)([\Gamma, X^{\mathrm{v}}])$. This may be easily computed if $[\Gamma, X^{\mathrm{v}}]$ is written as the sum of its vertical and horizontal parts, which are $X^c - X^h$ and $-X^h$. Thus

$$(L_\Gamma S)([\Gamma, X^{\mathrm{v}}]) = (L_\Gamma S)((X^c - X^h) - X^h) = (X^c - X^h) + X^h = X^c$$

and so $W = X^c$ as required.

Consideration of the special case of a geodesic spray is usually instructive regarding Lagrangian theory in general, and this is certainly true so far as symmetries are concerned. A Lie symmetry of the spray of a symmetric connection is a vector field which generates affine transformations of the base manifold M (on which the connection is defined). If the connection is metric, the spray is the Euler–Lagrange field of the kinetic energy function and the Noether symmetries are the isometries of the metric. Thus for example in $R^n = M$ the straight line geodesics have for Lie symmetries the infinitesimal generators of arbitrary affine transformations; the Euclidean metric provides a Lagrangian, the corresponding Noether symmetries being the infinitesimal generators of orthogonal transformations. But the Euclidean metric is not the only possibility for a Lagrangian: certainly, any non-singular symmetric constant bilinear form will do; it is significant that any such form may be generated by applying a non-orthogonal affine transformation to the Euclidean metric.

In the case of a Lie symmetry X of a second-order differential equation field Γ, it follows from the conditions $L_{X^c}S = 0$ and $[X^c, \Gamma] = 0$ that $L_{X^c}(L_\Gamma S) = 0$, and therefore the transformations of the one-parameter group generated by X^c preserve the direct sum decomposition of tangent spaces into horizontal and vertical subspaces. The horizontal subspaces for a geodesic spray are just those determined by the corresponding symmetric connection, which are preserved by the complete lifts of affine transformations and by those alone. Thus Lie symmetries generalise affine transformations. In the case of an Euler–Lagrange field Λ, if X is a Noether transformation then $L_{X^c}\omega = 0$; it follows that X^c is an isometry of the metric $(L_\Lambda S)\lrcorner\omega$. It is known (Yano and Ishihara 1973, p. 145) that for a (pseudo)-Riemannian manifold M, the complete lift of a vector field X on M generates isometries of the complete lift of the metric to $T(M)$ if and only if X generates isometries of M. Since the complete lift of the metric to $T(M)$ is just $(L_\Lambda S)\lrcorner\omega$, it is clear that Noether symmetries generalise isometries.

Consider, furthermore, a non-Noether Lie symmetry of an Euler–Lagrange field Λ, whose complete lift generates therefore a one-parameter group $\{\phi_t\}$ of transformations of $T(M)$ which leave the vector field Λ invariant. The two-form $\phi_t^*\omega$ is therefore distinct from ω, and is easily seen to satisfy all the conditions required for it to come from a Lagrangian for Λ (partly because, being a complete lift, ϕ_t leaves S invariant). (Prince, on whose paper (Prince 1983) these remarks are based, works with the Lie derivative rather than the pull-back of ω; it is not clear that the Lie derivative will have maximal rank, though it will satisfy the other conditions, so it seems preferable to use the pull-back.) In fact, if $\omega = \mathrm{d}(\mathrm{d}L\circ S)$ then $\phi_t^*\omega = \mathrm{d}(\phi_t^*(\mathrm{d}L\circ S)) = \mathrm{d}(\mathrm{d}(\phi_t^*L)\circ S)$, and so ϕ_t^*L is an alternative Lagrangian function for Λ, distinct from L (though possibly just a constant multiple of it). This construction generalises the use of a non-orthogonal affine transformation to generate an alternative Lagrangian for the straight line geodesic spray.

It will be apparent that the same conclusion will not hold in general for a non-Cartan dynamical symmetry, since it will not necessarily be the case that the transformations it generates will preserve the decomposition into horizontal and vertical subspaces. If W is a non-Cartan dynamical symmetry of the Euler–Lagrange field Λ, and $\{\phi_t\}$ the one-parameter group of transformations of $T(M)$ it generates, then $\phi_t^*\omega$ will satisfy all the conditions necessary for it to be derivable from a Lagrangian except those relating to the special properties of vertical subspaces. Now ω will be closed, and so $\phi_t^*\omega$ will also be closed; the additional restriction to be put on W is that $S\lrcorner\phi_t^*\omega$ is symmetric, which entails that $(L_W S)\lrcorner\omega$ should be symmetric, which is the condition found by Prince.

7. Formulae

Take coordinates q^a on M, and correspondingly q^a, u^a on $T(M)$ ($a = 1, 2, \ldots, m = \dim(M)$, with range and sum conventions in force). If $X = \xi^a\partial/\partial q^a$ is a vector field on M then

$$X^{\mathrm{v}} = \xi^a\frac{\partial}{\partial u^a} \qquad X^{\mathrm{c}} = \xi^a\frac{\partial}{\partial q^a} + u^b\frac{\partial\xi^a}{\partial q^b}\frac{\partial}{\partial u^a}.$$

The dilation field Δ is given by

$$\Delta = u^a\partial/\partial u^a$$

and the type (1, 1) tensor field S by

$$S = (\partial/\partial u^a)\otimes \mathrm{d}q^a.$$

A second-order differential equation field Γ takes the form

$$\Gamma = u^a\partial/\partial q^a + f^a\partial/\partial u^a \qquad f^a = f^a(q, u).$$

Its integral curves are the solutions of

$$\dot q^a = u^a \qquad \dot u^a = f^a(q, \dot q) \qquad \text{or} \qquad \ddot q^a = f^a(q, \dot q).$$

For any second-order differential equation field Γ,

$$L_\Gamma S = \left[\Gamma, \frac{\partial}{\partial u^a}\right] \otimes \mathrm{d}q^a + \frac{\partial}{\partial u^a} \otimes \mathrm{d}u^a = -\left(\frac{\partial}{\partial q^a} - 2A^b_a \frac{\partial}{\partial u^b}\right) \otimes \mathrm{d}q^a + \frac{\partial}{\partial u^a} \otimes \mathrm{d}u^a$$

where I have followed Sarlet (1982) in putting $A^b_a = -\frac{1}{2}\partial f^b/\partial u^a$. Thus

$$P = \left(\frac{\partial}{\partial q^a} - A^b_a \frac{\partial}{\partial u^b}\right) \otimes \mathrm{d}q^a \qquad Q = \frac{\partial}{\partial u^a} \otimes (A^a_b \mathrm{d}q^b + \mathrm{d}u^a).$$

The vector fields $H_a = \partial/\partial q^a - A^b_a \partial/\partial u^b$ form a basis for the horizontal subspace at each point (and of course the vector fields $V_a = \partial/\partial u^a$ form a basis for the vertical subspace at each point). The basis of one-forms dual to the basis $\{H_a, V_a\}$ is $\{\mathrm{d}q^a, \theta^a\}$ where $\theta^a = A^a_b \mathrm{d}q^b + \mathrm{d}u^a$. Rewriting $L_\Gamma S, P$ and Q in terms of these bases one finds

$$L_\Gamma S = -H_a \otimes \mathrm{d}q^a + V_a \otimes \theta^a \qquad P = H_a \otimes \mathrm{d}q^a \qquad Q = V_a \otimes \theta^a.$$

The Lie derivatives of these basis vector fields and forms with respect to Γ, expressed in terms of the basis fields themselves, are

$$[\Gamma, H_a] = A^b_a H_b + \Phi^b_a V_b \qquad [\Gamma, V_a] = -H_a + A^b_a V_b$$

$$L_\Gamma \mathrm{d}q^a = -A^a_b \mathrm{d}q^b + \theta^a \qquad L_\Gamma \theta^a = -\Phi^a_b \mathrm{d}q^b - A^a_b \theta^b$$

where (again following Sarlet)

$$\Phi^b_a = B^b_a - A^c_a A^b_c - \Gamma(A^b_a) \qquad \text{and} \qquad B^b_a = -\partial f^b/\partial q^a.$$

The two-form $\mathrm{d}(\mathrm{d}L \circ S)$, when expressed in terms of the one-form basis $\{\mathrm{d}q^a, \theta^a\}$ appropriate to its Euler–Lagrange field, takes the form

$$\mathrm{d}(\mathrm{d}L \circ S) = \alpha_{ab}\, \mathrm{d}q^a \wedge \theta^b \qquad \alpha_{ab} = -\partial^2 L/\partial u^a \partial u^b.$$

Conversely, any two-form ω such that, for a given second-order differential equation field Γ, $S \lrcorner \omega$ and $(L_\Gamma S) \lrcorner \omega$ are symmetric, necessarily takes the form

$$\omega = \alpha_{ab}\, \mathrm{d}q^a \wedge \theta^b \qquad \text{where } \alpha_{ab} = \alpha_{ba}.$$

The Helmholtz conditions arise as follows:

$$(L_\Gamma \omega)(H_a, H_b) = 0 \qquad \alpha_{ac} \Phi^c_b = \alpha_{bc} \Phi^c_a$$

$$(L_\Gamma \omega)(H_a, V_b) = 0 \qquad \Gamma(\alpha_{ab}) = \alpha_{ac} A^c_b + \alpha_{bc} A^c_a$$

$$(L_\Gamma \omega)(V_a, V_b) = 0 \qquad \text{(identity, in virtue of symmetry of } \alpha_{ab})$$

$$(i_{H_a} \mathrm{d}\omega)(V_b, V_c) = 0 \qquad \partial\alpha_{ab}/\partial u^c = \partial \alpha_{ac}/\partial u^b.$$

The further condition on $\mathrm{d}\omega$ which arises if one assumes that $S \lrcorner i_H \mathrm{d}\omega$ is symmetric is

$$H_b(\alpha_{ac}) + \alpha_{bd} V_a(A^d_c) = H_c(\alpha_{ab}) + \alpha_{cd} V_a(A^d_b)$$

which may be written (cf Sarlet 1982, theorem 5)

$$\frac{\partial \alpha_{ac}}{\partial q^b} + \frac{1}{2} \frac{\partial}{\partial u^a}\left(\alpha_{cd} \frac{\partial f^d}{\partial u^b}\right) = \frac{\partial \alpha_{ab}}{\partial q^c} + \frac{1}{2} \frac{\partial}{\partial u^a}\left(\alpha_{bd} \frac{\partial f^d}{\partial u^c}\right).$$

A type (1, 1) tensor field T takes the form

$$T = \kappa^a_b H_a \otimes \mathrm{d}q^b + \lambda^a_b H_a \otimes \theta^b + \mu^a_b V_a \otimes \mathrm{d}q^b + \nu^a_b V_a \otimes \theta^b.$$

If $\omega = \alpha_{ab}\,\mathrm{d}q^a \wedge \theta^b$ then $T \lrcorner \omega$ is symmetric if and only if

$$\alpha_{ac}\kappa^c_b + \alpha_{bc}\nu^c_a = 0 \qquad \alpha_{ac}\lambda^c_b = \alpha_{bc}\lambda^c_a \qquad \alpha_{ac}\mu^c_b = \alpha_{bc}\mu^c_a$$

or in the obvious matrix notation (with T meaning transpose)

$$\alpha\kappa + (\alpha\nu)^{\mathrm{T}} = 0 \qquad \alpha\lambda = (\alpha\lambda)^{\mathrm{T}} \qquad \alpha\mu = (\alpha\mu)^{\mathrm{T}}.$$

Pre- and post-multiplication by P or Q picks out the various components of T; in particular

$$Q \circ T \circ P = \mu^a_b V_a \otimes \mathrm{d}q^b.$$

The Lie derivatives of the basis tensor fields are

$$L_\Gamma(H_a \otimes \mathrm{d}q^b) = (A^c_a\delta^b_d - \delta^c_a A^b_d)H_c \otimes \mathrm{d}q^d + H_a \otimes \theta^b + \Phi^c_a V_c \otimes \mathrm{d}q^b$$
$$L_\Gamma(H_a \otimes \theta^b) = -\Phi^b_c H_a \otimes \mathrm{d}q^c + (A^c_a\delta^b_d - \delta^c_a A^b_d)H_c \otimes \theta^d + \Phi^c_a V_c \otimes \theta^b$$
$$L_\Gamma(V_a \otimes \mathrm{d}q^b) = -H_a \otimes \mathrm{d}q^b + (A^c_a\delta^b_d - \delta^c_a A^b_d)V_c \otimes \mathrm{d}q^d + V_a \otimes \theta^b$$
$$L_\Gamma(V_a \otimes \theta^b) = -H_a \otimes \theta^b - \Phi^b_c V_a \otimes \mathrm{d}q^c + (A^c_a\delta^b_d - \delta^c_a A^b_d)V_c \otimes \theta^d.$$

In particular

$$L_\Gamma(L_\Gamma S) = -2H_a \otimes \theta^a - 2\Phi^a_b V_a \otimes \mathrm{d}q^b$$

and so if $\Phi^{(0)} = -\frac{1}{2}Q \circ L_\Gamma(L_\Gamma S) \circ P$ then

$$\Phi^{(0)} = \Phi^a_b V_a \otimes \mathrm{d}q^b.$$

The $\Phi^{(k)}$ are defined inductively; setting

$$\Phi^{(k)} = {}^{(k)}\Phi^a_b V_a \otimes \mathrm{d}q^b$$

one finds that

$$L_\Gamma\Phi^{(k)} = (\Gamma({}^{(k)}\Phi^a_b) + A^a_c\,{}^{(k)}\Phi^c_b - {}^{(k)}\Phi^a_c A^c_b)V_a \otimes \mathrm{d}q^b - {}^{(k)}\Phi^a_b(H_a \otimes \mathrm{d}q^b - V_a \otimes \theta^b).$$

Defining matrices ${}^{(k)}\Phi$ inductively by

$${}^{(k+1)}\Phi = \Gamma({}^{(k)}\Phi) + [A, {}^{(k)}\Phi] \qquad {}^{(0)}\Phi = \Phi$$

(the bracket indicating the matrix commutator) one then has

$$\Phi^{(k+1)} = {}^{(k+1)}\Phi^a_b V_a \otimes \mathrm{d}q^b$$

and the symmetry of $\Phi^{(k)} \lrcorner \omega$ amounts to

$$\alpha\,{}^{(k)}\Phi = (\alpha\,{}^{(k)}\Phi)^{\mathrm{T}}$$

(cf Sarlet 1982, theorem 7).

A type (1, 1) tensor field R which preserves horizontal and vertical subspaces and acts identically in each takes the form

$$R = \rho^a_b(H_a \otimes \mathrm{d}q^b + V_a \otimes \theta^b).$$

Its Lie derivative is given by

$$L_\Gamma R = (\Gamma(\rho^a_b) + A^a_c\rho^c_b - \rho^a_c A^c_b)(H_a \otimes \mathrm{d}q^b + V_a \otimes \theta^b) + (\Phi^a_c\rho^c_b - \rho^a_c\Phi^c_b)V_a \otimes \mathrm{d}q^b.$$

Thus necessary and sufficient conditions for $L_\Gamma R = 0$ are

$$\Gamma(\rho) + [A, \rho] = 0 \qquad [{}^{(0)}\Phi, \rho] = 0.$$

(Notice that by differentiating the last equation one obtains

$$\Gamma({}^{(0)}\Phi)\rho + {}^{(0)}\Phi\Gamma(\rho) - \Gamma(\rho){}^{(0)}\Phi - \rho\Gamma({}^{(0)}\Phi) = 0$$

whence

$$\Gamma({}^{(0)}\Phi)\rho - {}^{(0)}\Phi[A, \rho] + [A, \rho]{}^{(0)}\Phi - \rho\Gamma({}^{(0)}\Phi) = 0 \qquad \text{or} \qquad [{}^{(1)}\Phi, \rho] = 0,$$

and so on.)

The conditions for a vector field $W = \xi^a H_a + \eta^a V_a$ to commute with Γ are

$$\eta^a = \Gamma(\xi^a) + A^a_b \xi^b \qquad \Gamma(\eta^a) + A^a_b \eta^b + \Phi^a_b \xi^b = 0$$

or in terms of ξ^a alone

$$\Gamma^2(\xi^a) + 2A^a_b\Gamma(\xi^b) + B^a_b\xi^b = 0.$$

The further condition that $(L_W S) \lrcorner \omega$ should be symmetric is obtained as follows:

$$L_W S = -\frac{\partial\xi^a}{\partial u^b} H_a \otimes \mathrm{d}q^b + \left(H_b(\xi^a) + \frac{\partial A^a_c}{\partial u^b}\xi^c - \frac{\partial\eta^a}{\partial u^b}\right) V_a \otimes \mathrm{d}q^b + \frac{\partial\xi^a}{\partial u^b} V_a \otimes \theta^b.$$

But

$$\begin{aligned}\frac{\partial\eta^a}{\partial u^b} &= V_b(\Gamma(\xi^a)) + \frac{\partial A^a_c}{\partial u^b}\xi^b + A^a_c\frac{\partial\xi^c}{\partial u^b}\\ &= [V_b, \Gamma](\xi^a) + \Gamma\left(\frac{\partial\xi^a}{\partial u^b}\right) + \frac{\partial A^a_c}{\partial u^b}\xi^b + A^a_c\frac{\partial\xi^c}{\partial u^b}\\ &= H_b(\xi^a) - A^c_b\frac{\partial\xi^a}{\partial u^c} + \Gamma\left(\frac{\partial\xi^a}{\partial u^b}\right) + \frac{\partial A^a_c}{\partial u^b}\xi^b + A^a_c\frac{\partial\xi^c}{\partial u^b}\end{aligned}$$

and therefore

$$H_b(\xi^a) + \frac{\partial A^a_c}{\partial u^b}\xi^c - \frac{\partial\eta^a}{\partial u^b} = -\left(\Gamma\left(\frac{\partial\xi^a}{\partial u^b}\right) + A^a_c\frac{\partial\xi^c}{\partial u^b} - \frac{\partial\xi^a}{\partial u^c}A^c_b\right).$$

The symmetry conditions are therefore

$$\alpha_{ac}\partial\xi^c/\partial u^b = \alpha_{bc}\partial\xi^c/\partial u^a$$

$$\alpha_{ac}\left(\Gamma\left(\frac{\partial\xi^c}{\partial u^b}\right) + A^c_d\frac{\partial\xi^d}{\partial u^b} - \frac{\partial\xi^c}{\partial u^d}A^d_b\right) = \alpha_{bc}\left(\Gamma\left(\frac{\partial\xi^c}{\partial u^a}\right) + A^c_d\frac{\partial\xi^d}{\partial u^a} - \frac{\partial\xi^c}{\partial u^d}A^d_a\right).$$

These are not, however, independent: the second is a consequence of the first and the Helmholtz condition $\Gamma(\alpha_{ab}) = \alpha_{ac}A^c_b + \alpha_{bc}A^c_a$. The required condition is thus (cf Prince 1983, theorem 3)

$$\alpha_{ac}\partial\xi^c/\partial u^b = \alpha_{bc}\partial\xi^c/\partial u^a.$$

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